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LETTER TO THE EDITOR

A new integrable modified Korteweg–de Vries equation with one half degree of nonlinearity

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Abstract. An alternative generalized equation which incorporates the modified Korteweg-de Vries (MKdV) and KdV equations is proposed and from it a new integrable system of the MKdV type is found.

Integrable systems play a fundamental role in nonlinear science. Since the discovery of the complete integrability of the KdV equation [1], a special interest has been growing in the exactly solvable systems. In this letter, we report a new integrable nonlinear evolution equation which has the form

$$16u_t + u_{xxx} + 30u^{1/2}u_x = 0. \tag{1}$$

This equation is a special case of the following generalized MKdV equation

$$n^2u_t + u_{xxx} + (n+1)(n+2)u^{2/n}u_x = 0 \tag{2}$$

where n is a positive integer. For $n=4$, (2) gives (1). Clearly, (2) also incorporates the MKdV equation ($n=1$) and the KdV equation ($n=2$). However, unlike the generalized KdV equation [2]

$$u_t + u_{xxx} + u^n u_x = 0 \tag{3}$$

which takes the KdV equation as the basic equation, (2) takes the MKdV equation as the base.

In the following, we investigate the integrability of (2) through the Painlevé method [3]. The substitution of $u = q^n$ leads to the equation

$$n^2q^2q_t + (n-1)(n-2)q^3 + 3(n-1)qq_xq_{xx} + q^2q_{xxx} + (n+1)(n+2)q^4q_x = 0. \tag{4}$$

We assume that

$$q = \phi^\alpha \sum_{j=0}^{\infty} V_j\phi^j \tag{5}$$

where $V_j = V_j(x, t)$, ϕ is a singular manifold. Substituting (5) into (2), it is found that

$$\alpha = -1 \qquad V_0^2 = -\phi_x^2 \tag{6}$$

and the recursion relation takes the form

$$(j+1)[j-(n+2)][j-(2n+2)]V_0^2\phi_x^3V_j = F(V_{j-1}, \ldots V_0; \phi_t, \phi_x, \ldots). \tag{7}$$

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Since the detailed form of F is very complicated, it will be presented elsewhere [4]. It is clear from (7) that the resonances occur at

$$j=-1 \qquad n+2 \qquad 2n+2. \tag{8}$$

From (8) we find that

$$j=0:\ V_0^2=-1 \tag{9}$$

$$j=1:\ V_1=0 \tag{10}$$

$$j=2:\ V_2=V_0\psi_t/6 \tag{11}$$

$$j=3:\ (n-1)V_3=0 \tag{12}$$

$$j=4:\ (n-1)(n-2)(10V_4+7V_0V_2^2)=0 \tag{13}$$

$$j=5:\ 6(n-3)(2n-3)V_5+n^2V_{2t}=0 \tag{14}$$

$$j=6:\ (n-2)(n-4)(14V_6+31V_2^3/5)=0 \tag{15}$$

$$j=7:\ 10(2n-5)(n-5)V_7+(76n^2-327n+302)V_0V_2V_5=0 \tag{16}$$

$$\begin{aligned} j=8:\ &18(n-3)(n-6)V_8+n^2V_{5t}+(191n^2+141n-2762)3V_0V_2^4/100\\ &+6(4n^2-15n+2)V_0V_2V_6=0 \end{aligned} \tag{17}$$

$$\begin{aligned} j=10:\ &-22(n-4)(n-8)V_{10}+n^2(2V_0V_2V_5-V_7)_t+(-58n^2+246n-206)V_0V_5^2\\ &+(-32n^2+114n+236)V_0V_2V_8+(-641n^2+3117n+2078)3V_2^5/100\\ &+4(-8n^2+60n-67)V_2^2V_6=0 \end{aligned} \tag{18}$$

where we have used the reduced ansatz [5]

$$\phi=x+\psi(t) \qquad \phi_x=1 \qquad V_j=V_j(t).$$

When $n=1$ and $n=2$, the compatability conditions at $j=n+2$ and $j=2n+2$ are satisfied identically and the MKdV and KdV equations possess the Painlevé property. This is a well known fact [3]. However, the result that (2) with $n=4$, i.e. (1), also possesses the Painlevé property, and is thus identified as integrable, is new. This is easy to see from (15) and (18) since the resonances occur at $j=6$ and $j=10$ respectively. The integrability of (2) with general n is difficult to investigate through the Painlevé method since (7) becomes very complicated for large n.

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